SOLUTION OF A PLANE, AXISYMMETRIC AND THREE-DIMENSIONAL SINGLE-PHASE STEFAN PROBLEM

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Using a superposition method we construct a solution of the multidimensional problem of the steady-state fusion regime of semibounded solids. The solution of the problem is reduced to a generalized Fredholm integral equation of the first kind. A method is given for solving the integral equation for plane problems by converting to a linear system of algebraic equations.

The problem of the temperature distribution in a solid when the melted phase moves away from the surface of the solid is investigated, and the law of motion of the fusion boundary is a variant of the Stefan problem. Such a problem arises in the study of the laws of fusion of bodies subjected to the flow of a hot gas or liquid past them.

One-dimensional nonsteady and steady Stefan problems have been sufficiently completely investigated. A review of these studies and the principal results are given in [1]. Considerable mathematical difficulties arise in the solution of the non-one-dimensional Stefan problems. At present there are no analytical solutions for multidimensional problems with phase increments; only numerical methods of solution of similar problems have been developed [2].

1. We assume that the body fuses along its surfaces. The temperature U on the surface Σ of fusion of the solid equals the temperature of the phase transition, which will be assumed equal to zero:

$$U(x,\ y,\ z,\ t)|_{\Sigma}=0. \tag{1.1}$$

In the interior points of the solid the temperature can be less than or equal to the temperature of the phase transition:

$$U(x,\ y,\ z,\ t)\leqslant 0. \tag{1.2}$$

The heat q supplied to the body from the external hot flow is used in heating the body to the melting point and for the transition of the solid phase into the liquid phase

$$q=QV_n-\lambda\partial U/\partial n|_{\Sigma}. \tag{1.3}$$

For active fusion, the velocity V_n is positive, i.e.,

$$V_n\geqslant 0. \tag{1.4}$$

We add the following initial condition to the boundary conditions (1.1) and (1.3):

$$U(x,\ y,\ z,\ 0)=f(x,\ y,\ z). \tag{1.5}$$

We write the heat equation in the form

$$\frac{\partial U}{\partial t}=a\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right). \tag{1.6}$$

We consider the problem of the steady-state fusion regime for a solid. We assume that for a time sufficiently far from the initial time there exists a noninertial moving coordinate system (x*, y*, z*) in

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which the heat flux q, the form of the fusion surface Σ, and its velocity of translational motion V_0, and also the temperature of the points of the solid, do not depend on the time. It is evident that a steady-state fusion regime in general is possible only for semibounded tapered bodies. Dropping the superscript asterisks (x*, y*, z*), the problem takes the form

$$a\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)+V_0\frac{\partial U}{\partial z}=0, \tag{1.7}$$

$$U(x,\ y,\ z)|_\Sigma=0,\quad U(x,\ y,\ z)\leqslant 0, \tag{1.8}$$

$$q=QV_n-\lambda\partial U/\partial n|_\Sigma,\quad V_n\geqslant 0. \tag{1.9}$$

Here the direction of the z axis coincides with the direction of translational motion of the moving coordinate system. Instead of initial condition (1.5) we set the condition at infinity

$$U(x,\ y,\ z)\to U_\infty\leqslant 0 \quad \text{as} \quad r_M\to\infty. \tag{1.10}$$

Problem (1.7)-(1.10) has two formulations: a) based on a given form of the fusion surface Σ, its velocity of motion V_0, and temperature at infinity U_∞, we must determine the temperature field at the solid phase and the heat flux q; b) based on the given heat flux q and temperature at infinity U_∞, we must determine the temperature field at the solid phase, the form of the fusion surface Σ, and its velocity of translational motion V_0.

2. Solution for steady-state fusion regime. We first consider the plane problem in the (y, z) coordinate system. We introduce the notation

$$\xi=z\sin\theta-y\cos\theta. \tag{2.1}$$

The geometrical sense of the quantity ξ is the distance from the point (z, y) to the line ξ = 0. The positive direction of the line ξ = 0 is chosen so that ξ > 0 on the right, and ξ < 0 on the left.

We will seek a particular solution of Eq. (1.7) in the form

$$U(z,\ y)=U(\xi)=C_1\exp(-\alpha\xi)+C_2. \tag{2.2}$$

Substituting (2.2) into (1.7), we obtain

$$\alpha=(V_0\sin\theta)/a. \tag{2.3}$$

Thus, for the particular solution we have the equation

$$U=C_1\exp(-\xi V_0\sin\theta/a)+C_2. \tag{2.4}$$

Let $C_1 = C(\theta)$; then, using the superposition principle, we can write the solution of Eq. (1.7) in the form

$$U=U_\infty\left[1-\int_{\alpha_1}^{\alpha_2}C(\theta)\exp[-\eta(\theta)]\,d\theta-\sum_{i=1}^{n}C_i\exp(-\eta_i)\right],$$
$$\eta(\theta)=\sin\theta\,(z\sin\theta-y\cos\theta)\,V_0/a. \tag{2.5}$$

For θ = 0 or θ = π the particular solution (2.4) degenerates to a constant; therefore, the values θ = 0 and θ = π can be eliminated from the solution (2.5). Thus, the limits of integration α_1 and α_2, and also the angles θ and θ_i in the solution (2.5) should not assume the values 0 and π. And since $\eta(\theta)=\eta(\pi+\theta)$, without loss of generality in the investigations we can assume that the limits of variation of the angles α_1, α_2, and θ_i are contained on the interval (0, π).

Satisfying condition (1.8), we find the equation for the fusion surface

$$\Phi(y,\ z)=\int_{\alpha_1}^{\alpha_2}C(\theta)\exp(-\eta(\theta))\,d\theta+\sum_{i=1}^{n}C_i\exp(-\eta_i)=1. \tag{2.6}$$

We obtain for the heat flux from (1.9) the equation

$$QV_0\Phi_z/\sqrt{\Phi_y^2+\Phi_z^2}+\lambda U_\infty\sqrt{\Phi_y^2+\Phi_z^2}=-q \quad \text{for} \quad \Phi=1. \tag{2.7}$$

We convert to the construction of a solution of the problem in the formulation a). In this case the equation of the surface Σ is assumed to be known and have the parametric form

$$z=f_1(s),\quad y=f_2(s). \tag{2.8}$$

Substituting (2.8) into (2.6), we arrive at the generalized Fredholm integral equation of the first kind in the functions $C(\theta)$ and C_i:

$$\int_{\alpha_1}^{\alpha_2} C(\theta) \exp\left[-\frac{V_0}{a} \sin\theta\,(f_1(s)\sin\theta - f_2(s)\cos\theta)\right] d\theta + \sum_{i=1}^{n} C_i \exp\left[-\frac{V_0}{a} \sin\theta_i\,(f_1(s)\sin\theta_i - f_2(s)\cos\theta_i)\right] = 1. \tag{2.9}$$

From (2.7) we find the heat flux q, and from (2.5) we find the temperature field. If the problem is solved in the formulation b), then in order to find C_i, $C(\theta)$, and V_0 from (2.7) we arrive at the nonlinear integral equation (2.7), where Φ_y and Φ_z have the form

$$\Phi_y = \int_{\alpha_1}^{\alpha_2} C(\theta)\sin\theta\cos\theta\exp[-\eta(\theta)]\,d\theta + \sum_{i=1}^{n} C_i \sin\theta_i\cos\theta_i\exp(-\eta_i),$$
$$\Phi_z = -\int_{\alpha_1}^{\alpha_2} C(\theta)\sin^2\theta\exp[-\eta(\theta)]\,d\theta - \sum_{i=1}^{n} C_i\sin^2\theta_i\exp(-\eta_i). \tag{2.10}$$

Here it is necessary to give the position of the front point of the surface Σ or any other point. We convert to the study of the properties of the solution (2.5). In subsequent investigations we will assume that the angles θ_i are arranged in an increasing sequence $0 < \theta_i < \theta_2 < \ldots < \theta_n < \pi$. We prove that in (2.5) the solution in the form of a finite sum (when $C(\theta) = 0$) for $C_i > 0$ has the properties A_1—A_5.

A_1. The two lines $\eta_1 = \ln C_1$ and $\eta_n = \ln C_n$ are asymptotes to the fusion surface Σ.

A_2. The surface Σ intersects none of the lines $\eta_i = \ln C_i$ ($i = 1, \ldots, n$), and approaches its asymptotes to the right.

A_3. Any half-line l drawn from a point of the surface Σ on the right parallel to the z axis belongs entirely to the region of the fused body. For motion along l in the direction of increasing z, the temperature U monotonically varies from 0 to U_∞.

A_4. At each point of the surface Σ the angle between the inner normal and the z axis is acute, i.e., inequality (2.8) is satisfied.

A_5. We consider the two solutions U_1 and U_2 from (2.5), corresponding to the two sets $\{C_i'\}$ and $\{C_i''\}$. Let $C_i' = C_i''$, $\theta_i' = \theta_i''$ for $i \neq k$ and $C_k' > C_k''$, $\theta_k' = \theta_k''$. Then on the (z, y) plane we have $U_1 > U_2$ and the fusion surface Σ_1 will be located to the right of the surface Σ_2. In this case the surfaces Σ_1 and Σ_2 do not intersect anywhere, and they have the same asymptotes.

B_1. If $C_2 < 0$ or $C_{n-1} < 0$, then the surface Σ approaches from the left the asymptotes $\eta_1 = \ln C_1$ or $\eta_n = \ln C_n$, respectively.

We prove the property A_1. On the line $\eta_1 = \ln C_1$ the function $\Phi(y, z) = \sum_{i=1}^{n} C_i \exp(-\eta_i)$ takes the values

$$\Phi(y, z) = 1 + \sum_{i=2}^{n} C_i \exp(-\eta_i) > 1 \text{ when } \eta_1 = \ln C_1. \tag{2.11}$$

Since $0 < \theta_1 < \theta_i < \pi$ ($i = 2\ldots n$), for motion along the line $\eta_1 = \ln C_1$ in the positive direction the variables $\eta_2 \ldots \eta_n$ increase without bound and $\Phi(y, z) \to 1$. Hence we obtain the proof of property A_1. As an explanation we note that for $n \geq 3$ the line $\eta_2 = \ln C_2$ cannot be an asymptote of the surface Σ because of the following reasons: on the line $\eta_2 = \ln C_2$ for $\Phi(y, z)$ we will have the equation

$$\Phi(y, z) = 1 + C_1\exp(-\eta_1) + \sum_{i=3}^{n} C_i \exp(-\eta_i). \tag{2.12}$$

The inequalities $0 \leq \theta_1 < \theta_2 < \theta_i \leq \pi$ insure that for motion along the line $\eta_2 = \ln C_2$ in the positive direction the variables $\eta_3 \ldots \eta_n$ increase without bound, and η_1 decreases without bound. For motion along the line $\eta_2 = \ln C_2$ in the negative direction the variables $\eta_3 \ldots \eta_n$ will decrease without bound, and η_1 will increase without bound, so that everywhere along the line $\eta_2 = \ln C_2$ we will have $\Phi(y, z) > 1$. Hence, in particular, it follows that the surface Σ nowhere intersects the line $\eta_2 = \ln C_2$. The surface Σ also does not intersect the asymptotes, which follows from the inequality in (2.11), and since on each line $\eta_i = \ln C_i$ the inequality $\Phi(y, z) > 1$ is satisfied, we obtain the proof of property A_2. Let $C_2 < 0$. For motion along the line $\eta_1 = \ln C_1$ in the positive direction the variables $\eta_3 \ldots \eta_n$ increase more rapidly than η_2 (since $0 < \theta_2 < \theta_3 < \ldots < \theta_n \leq \pi$); therefore, for a sufficient distance from the line $\eta_2 = \ln(-C_2)$, when $\eta_i \gg \eta_2$ ($i = 3 \ldots n$) the sign of the sum in (2.11) will be determined by the sign of the principal term $C_2 \exp(-\eta_2) < 0$. Hence we obtain that for $\eta_1 = \ln C_1$ and $\eta_i \to \infty$ ($i = 2 \ldots n$), $C_2 < 0$ the function $\Phi(y, z) < 1$, i.e., the surface Σ approaches the asymptote $\eta_1 = \ln C_1$ from the left (property B_1).

For proof of property A_3 we differentiate $\Phi(y, z)$ with respect to the coordinate z:

$$\Phi_z = -\frac{V_0}{a} \sum_{i=1}^{n} C_i \sin^2\theta_i \exp(-\eta_i) < 0. \tag{2.13}$$

Along the semibounded line l, drawn from a point of the surface Σ parallel to the z axis, with increasing z, the variables η_i ($i = 1 \ldots n$) increase without bound, and it follows from the inequality $\Phi_z < 0$ that along l the function $\Phi(y, z)$ varies monotonically from 1 on Σ to 0 at infinity. Hence we have the proof of property A_3. As a corollary of property A_3 we obtain the proof of property A_4, since if the angle between the inner normal and the z axis is obtuse, then the half-line l will not belong entirely to the region of the fused body.

We write the functions Φ_1 and Φ_2 corresponding to the two solutions U_1 and U_2 from (2.5) in the form

$$\Phi_1 = \sum_{\substack{i=1 \\ i \neq k}}^{n} C_i \exp(-\eta_i) + C_k' \exp(-\eta_k),$$
$$\Phi_2 = \sum_{\substack{i=1 \\ i \neq k}}^{n} C_i \exp(-\eta_i) + C_k'' \exp(-\eta_k). \tag{2.14}$$

Let $C_k' > C_k''$; then $\Phi_1 > \Phi_2$ and on the basis of property A_3 the coordinates of the points M_1 and M_2, belonging to the surfaces Σ_1 and Σ_2, respectively, have the following property: for $y_1 = y_2$ it is necessary that $z_1 > z_2$, which also proves property A_5.

We investigate the solution from (2.5) in the form of the sum of the integral with finite sum. In this case $C(\theta)$ and C_i should be determined from the generalized Fredholm integral equation of the first kind

$$\int_{\alpha_1}^{\alpha_2} C(\theta) \exp(-\eta)\, d\theta + \sum_{i=1}^{n} C_i \exp(-\eta_i) = 1. \tag{2.15}$$

It is known that in Eq. (2.15) the unknown function $C(\theta)$ can have the form

$$C(\theta) = B(\theta) + \sum_{i=n}^{m+n} C_i \delta(\theta - \theta_i), \tag{2.16}$$

where $B(\theta)$ is a function that is summable and integrable in the sense of Lebesgue on the interval $[\alpha_1, \alpha_2]$ without singularities of δ-function type, $\delta(\theta - \theta_i)$ is the Dirac delta function. Substitution of (2.16) into (2.15) leads only to a change in the number of terms in the finite sum; therefore, below we assume that $C(\theta)$ in (2.5) and (2.15) is a function that is summable and integrable in the sense of Lebesgue on the interval $[\alpha_1, \alpha_2]$ without singularities of δ-function type.

We split the interval of integration in (2.5) and (2.15) into p small parts and we represent the integral equation (2.15) in the form of the finite sum

$$\sum_{k=1}^{p} C_k^* \exp[(-\eta(\theta)_k^*)]\, \Delta\theta_k + \sum_{i=1}^{n} C_i \exp[-\eta(\theta_i)] = 1. \tag{2.17}$$

The asterisk in (2.17) denotes that the mean value of the function is taken on the appropriate interval. Similarly to (2.17) we write the solution (2.5) in the form of a finite sum, and we can then use the properties A_1–A_5 and B_1.

If $\alpha_1 < \theta_1$ and $C_1^* > 0$, then according to property A_1, the line

$$\eta_1^* = \ln(C_1^* \Delta\theta_1) = \ln\left(\int_{\alpha_1}^{\alpha_1+\Delta\theta_1} C(\theta)\, d\theta\right) \tag{2.18}$$

is an asymptote to the surface (2.17). Converting to the limit in (2.18) for $\Delta\theta_1 \to 0$, we find that $C_1^* \Delta\theta_1 \to 0$ and one of the branches of the surface Σ will not have asymptotes, and the angle α_1 will be the limiting angle of the tangent to this branch, since the angle θ in the equation $\eta(\theta) = \text{const}$ equals the angle between this line and the z axis. Let $\theta_1 \le \alpha_1$ and $C_1 > 0$. Then from property A_1 it follows that in the limit as $\Delta\theta_1 \to 0$ the equation of the asymptote to surface (2.17) has the form

$$\eta_1 = \ln C_1. \tag{2.19}$$

Thus we have proved Theorem I:

If $\alpha_1 < \theta_1$ and $C(\alpha_1) > 0$, then one of the branches of the surface (2.15) does not have asymptotes, and the limiting value of the angle between the tangent to this branch and the z axis equals α_1. If $\theta_1 \le \alpha_1$, then the line (2.19) is an asymptote to the surface (2.15).

In a similar way we prove Theorem II:

if $\alpha_2 > \theta_n$ and $C(\alpha_2) > 0$, then one of the branches of the surface (2.15) does not have asymptotes, and the limiting value of the angle between the tangent to this branch and the z axis equals α_2. If $\theta_n = \alpha_2$, then the line $\eta_n = \ln C_n$ is an asymptote to the surface (2.15).

Now, using Theorems I and II we can determine the limits of integration α_1 and α_2 in (2.5) and (2.15) if $\alpha_1 < \theta_1$ and $\alpha_2 > \theta_n$, using the equation of the surface Σ in the form (2.8). When $\alpha_1 \ge \theta_1$ and $C_1 > 0$, as the lower limit of integration in (2.5) and (2.15) we can take θ_1, assuming the unknown function $C(\theta)$ to be equal to zero on the interval $[\theta_1, \alpha_1]$. We can proceed similarly in other cases: when $\alpha_2 \le \theta_n$ or when $\alpha_2 \le \theta_n$ and $\alpha_1 \ge \theta_1$.

We obtain the simplest solution of the problem of the fusion of an infinite wedge with apex angle $2\theta_0$ from (2.5) if we set $C_1 = C_2 = 1/2$, $C(\theta) = 0$:

$$U = U_\infty\left[1 - \exp\left(-\frac{V_0}{a}\, z\sin^2\theta_0\right)\operatorname{ch}\left(\frac{V_0}{2a}\, y\sin 2\theta_0\right)\right]. \tag{2.20}$$

This solution corresponds to the case in which for a steady-state process of fusion of a wedge the equation of its surface has the form

$$\frac{a}{V_0\sin^2\theta_0}\ln\operatorname{ch}\left(\frac{V_0}{2a}\, y\sin 2\theta_0\right) = z. \tag{2.21}$$

If the (xz) plane is a symmetry plane of the surface (2.15), then the coefficients C_i, the angles θ_i, and the function $C(\theta)$ should have the properties

$$C(\theta) = C(\pi - \theta),\quad C_i = C_{n-i+1},\quad \theta_i = \pi - \theta_{n-i+1}. \tag{2.22}$$

We proceed to the solution of the integral equation (2.15) for $C(\theta)$ and C_i. We consider (2.17) instead of (2.15). It is obvious that for each angle θ_i we find an angle θ_k^* such that θ_i and θ_k^* will belong to the same small interval, into which the interval of integration $[\alpha_1, \alpha_2]$ is divided. We introduce the notation

$$C_k^*\exp[-\eta(\theta_k^*)]\,\Delta\theta_k + C_i\exp[-\eta(\theta_i)] = B_i\exp[-\eta(\theta_i^*)]. \tag{2.23}$$

We note that the number of terms in the second sum of (2.17) is constant, and in the first sum it depends on the partition step of the interval $[\alpha_1, \alpha_2]$. Therefore, generally speaking, each term of the second sum will correspond to a term of the first sum according to the scheme (2.23). There exists some term from the first sum that does not have a corresponding term from the second sum in (2.17). For convenience we denote such terms from the first sum in (2.17) as

$$C_k^*\Delta\theta_k = B_k. \tag{2.24}$$

Now, using (2.23) and (2.24), Eq. (2.17) takes the form

$$\sum_{k=1}^{p} B_k\exp[-\eta(\theta_k^*)] = 1. \tag{2.25}$$

We recall that on the surface Σ the dependencies $\eta(\theta_k^*)$ have the form

$$\eta(\theta_k^*) = \frac{V_0}{a}\sin\theta_k^*\,[f_1(s)\sin\theta_k^* - f_2(s)\cos\theta_k^*]. \tag{2.26}$$

The problem of finding C_i and $C(\theta)$ reduces to the problem of finding the coefficients B_k of Eq. (2.25). Giving the parameter s, which appears in this equation, p different values, we obtain a closed system of linear algebraic equations for the coefficients B_k:

$$\sum_{k=1}^{p} B_k \exp[-\eta_r(\theta_k^*)] = 1, \quad \eta_r(\theta_k^*) = \eta(\theta_k^*) \text{ for } s = s_r \ (r = 1 \ldots p). \tag{2.27}$$

For the solvability of system (2.27) we require a nonzero determinant of this system, which is equivalent to the requirement that the homogeneous integral equation

$$\int_{\alpha_1}^{\alpha_2} C(\theta) \exp(-\eta)\, d\theta + \sum_{i=1}^{n} C_i \exp(-\eta_i) = 0 \tag{2.28}$$

has only the trivial solution

$$C(\theta) = 0, \ C_i = 0. \tag{2.29}$$

This requirement coincides with the well-known theorem of uniqueness of the solution of an integral equation. The kernel $\exp(-\eta)$ does not have a singularity; therefore, the system of equations (2.27) always has a unique solution. We assume that the coefficients B_k from system (2.27) have been found and it now remains to determine the coefficients C_i and C_k^*. According to (2.24), some of the coefficients C_k^* equal the ratio $B_k^*/\Delta\theta_k$. The remaining C_k^* and C_i should be determined from Eqs. (2.23). Before calculating these coefficients we must determine their number and the value of those angles θ_i that correspond to the coefficients C_i. To do this we must pay attention to the order of magnitude of B_k from (2.23) and (2.24). Since C_k^* and C_i are assumed to be finite, the coefficients B_k from (2.23) have a finite value, and the coefficients B_k from (2.24) are small and their magnitude depends on the step $\Delta\theta_k$. Furthermore, the number of finite B_k for a sufficiently small step is independent of the magnitude of this step. Thus the number of coefficients C_i equals the number of coefficients B_k, assuming a finite value for sufficiently small $\Delta\theta_k$. Based on the index of the final value of B_k we find the corresponding interval of variation of the angle θ. The mean value of θ in this interval can be taken equal to the value of the angle θ_i. It remains for us to find C_i and C_i^* from (2.23).

We note that C_i in (2.23) is a term that is independent of the step $\Delta\theta_i$, whereas the second term depends on this step. In connection with this, we divide the interval $[\alpha_1, \alpha_2]$ into small parts by two methods with steps $\Delta\theta_k'$ and $\Delta\theta_k''$. Then the final quantities B_k' and B_k'', determined from system (2.27) by two methods, should have close values, and the intervals corresponding to them should have a common part. According to (2.23) for B_i' and B_i'' we have the expressions

$$C_i \exp[-\eta(\theta_i)] + C_i^{*'} \exp[-\eta(\theta_i^{*'})]\, \Delta\theta_i' = B_i' \exp[-\eta(\theta_i^{*'})],$$

$$C_i \exp[-\eta(\theta_i)] + C_i^{*''} \exp[-\eta(\theta_i^{*''})]\, \Delta\theta_i'' = B_i'' \exp[-\eta(\theta_i^{*''})]. \tag{2.30}$$

In (2.30) we can take approximately

$$C_i^{*'} \approx C_i^{*''}, \quad \theta_i^{*'} \approx \theta_i^{*''}. \tag{2.31}$$

Taking the difference of the left and right sides of the two equations in (2.30), we obtain for C_i^* the expression

$$C_i^* \approx (B_i' - B_i'')/(\Delta\theta_i' - \Delta\theta_i''). \tag{2.32}$$

Eliminating C_i^* from (2.23) using (2.32) and assuming

$$\theta_i \approx \theta_i^*, \tag{2.33}$$

for finding C_i we arrive at the formula

$$C_i \approx B_i - C_i^* \Delta\theta_i. \tag{2.34}$$

It is obvious that the approximate equalities (2.31)-(2.34) with reduction in the partition of the interval $[\alpha_1, \alpha_2]$, when $\Delta\theta_i \to 0$, in the limit, become exact equalities. In the realization of the proposed numerical method for the coefficients C_i, the functions $C(\theta)$ and the angles θ_i, approximate values are found; however, the solution (2.5) will exactly satisfy the heat equation, and only the boundary condition on the fusion surface Σ from (1.8) will be satisfied approximately.

If in the integral equation (2.15) the coefficients C_i and the angles θ_i are known, then the solution of this equation can be sought in the form of an expansion of $C(\theta)$ in eigenfunctions of the kernel $\exp(-\eta)$. However, difficulties arise in the proof of the convergence of this method owing to the semiboundedness of the region of the fused body. It is interesting to note that the proposed method of solution of two-dimensional problems on the steady-state regime of fusion of a body from the point of view of the numerical realization is considerably simpler than the solution of a system of ordinary first-order differential equations for solution of the unsteady one-dimensional Stefan problems [3, 4].

3. Carrying out the same reasoning as for the plane case, we obtain for the three-dimensional problem a solution in the form

$$U = U_\infty \Big[1 - \int_0^{2\pi} d\varphi \int_{\alpha(\varphi)}^{\pi/2} C(\theta, \varphi) \exp(-\eta)\, d\theta - \sum_{i,\, j=1}^{i=m_1,\, n_1=j}\sum C_{ij} \exp(-\eta_{ij}) -$$
$$- \int_0^{2\pi} \sum_{j=1}^{n_2} C_{j0} \exp(-\eta_{j0})\, d\varphi - \sum_{i=1}^{m_2} \int_{\alpha(\varphi_i)}^{\pi/2} C_{0i} \exp(-\eta_{0i})\, d\theta\Big], \tag{3.1}$$
$$\eta(\theta, \varphi) = \frac{V_0}{a} \sin\theta\,(z \sin\theta - x\cos\theta\cos\varphi - y\cos\theta\sin\varphi),$$
$$\eta_{ij} = \eta(\theta_i, \varphi_j),\ \eta_{j0} = \eta(\theta_j, \varphi),\ \eta_{0i} = \eta(\theta, \varphi_i).$$

Let the fused body have a front point through which the z_1 axis is drawn parallel to the z axis. We denote by Γ the curve that is obtained for intersection of the fusion surface with the half-plane drawn through z_1 at an angle φ to the (xz) plane. Just as in the plane problem, we prove here that the angle $\alpha(\varphi)$, which is the lower limit in the integrals (3.1), equals the limiting value of the angle between the tangent to Γ and the z_1 axis. Furthermore, the solution (3.1) has properties similar to the properties A_1–A_5 and B_1 in the plane problem.

In (3.1) the unknowns are the functions $C(\theta, \varphi)$, $C_{j0}(\varphi)$, and $C_{0i}(\theta)$, the coefficients C_{ij}, and the angles θ_i, φ_j under each summation sign. We write the equation of the fusion surface Σ in parametric form

$$x = x(s_1, s_2),\quad y = y(s_1, s_2),\quad z = z(s_1, s_2). \tag{3.2}$$

Substitution of (3.2) and (3.1) into the boundary condition (1.8) leads to a Fredholm integral equation of the first kind in the unknowns C, C_{j0}, C_{0i}, C_{ij}, θ_i, and φ_j:

$$\int_0^{2\pi} d\varphi \int_{\alpha(\varphi)}^{\pi/2} C \exp(-\eta)\, d\theta + \int_0^{2\pi} \sum_{j=1}^{n_2} C_{j0} \exp(-\eta_{j0})\, d\varphi +$$
$$+ \sum_{i=1}^{m_2} \int_{\alpha(\varphi_i)}^{\pi/2} C_{0i} \exp(-\eta_{0i})\, d\theta + \sum_{i,\, j=1}^{i=m_1,\, j=n_1} C_{ij} \exp(-\eta_{ij}) = 1,\quad \pi/2 \geqslant \alpha \geqslant 0. \tag{3.3}$$

Detailed proofs and the solution of the integral equation (3.3) will be given in a subsequent paper. In conclusion we present a solution of a problem with axial symmetry. We introduce the notation

$$x = r\cos\psi,\ y = r\sin\psi. \tag{3.4}$$

Then we obtain from (3.1) for the variable η

$$\eta = \frac{V_0}{a} \sin\theta\,[z\sin\theta - r\cos\theta\cos(\varphi - \psi)]. \tag{3.5}$$

Substitution of (3.5) into (3.1) after a number of simplifications leads to the solution of the problem of the fusion of an axisymmetric body

$$U = U_\infty \Big[1 - \int_{\theta_0}^{\pi/2} A(\theta) \exp(-\eta_z) I_0(\eta_r) - \sum_{i=1}^{n} A_i \exp(-\eta_{zi}) I_0(\eta_{ri})\Big], \tag{3.6}$$
$$\eta_z = \frac{V_0}{a} z\sin^2\theta,\qquad \eta_r = \frac{V_0}{2a} r\sin 2\theta.$$

Here the function $I_0(z) = J_0(iz)$ is a Bessel function of the first kind of zero order with an imaginary argument [5]. The simplest solution of the problem from (3.6) for the fusion of a cone with apex angle $2\theta_0$ has the form

$$U = U_\infty \left[1 - A \exp\left(-\frac{V_0}{a} z\sin^2\theta_0\right) I_0\left(\frac{V_0}{2a} r\sin 2\theta_0\right)\right]. \tag{3.7}$$

NOTATION

U	is the temperature;
Σ	is the surface of the fused body;
(x, y, z)	are rectangular Cartesian coordinates;

t	is the time;
q	is the heat flux supplied from the external hot flow to the fused body;
Q	is the latent heat of the phase transformation;
V_n	is the velocity of the fused surface Σ;
n	is the normal to the surface Σ, directed toward the solid phase;
λ	is the thermal conductivity;
ρ	is the density;
C	is the specific heat;
$a = \lambda / c\rho$	is the thermal diffusivity;
V_0	is the velocity;
U_∞	is the temperature at points of the solid infinitely distant from Σ;
r_M	is the shortest distance from a point of the body to Σ;
C_1 and C_2	are the constants of integration;
ξ and η	are the self-similar variables;
s	is the parameter in the equation of the fusion surface;
Φ_y and Φ_z	are the partial derivatives of the function Φ with respect to y and z;
$\delta(\theta)$	is the generalized Dirac delta function;
α and β	are the angles between the normal to the plane $\xi = 0$ and the x and y axes, respectively;
$J_0(iy)$	is a Bessel function of the first kind of zero order with an imaginary argument;
θ	is the angle between the plane $\xi = 0$ and the z axis.

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